Note on Inventory Management

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1 Economic Order Quantity Model

1.1 Economic Order Quantity (EOQ)

Definition 1.1

1. *D: Constant demand's rate per unit time*
2. *Q: Fixed quantities per order, and $T = Q/D$ is the time between two successive replenishments as a reorder interval.*
3. *K: Fixed set-up cost per order*
4. *h: inventory holding cost*
5. *$I(t)$: inventory level at time t*

Assumption 1.1

1. *The supplier has an unlimited quantity of the product.*
2. *The lead time is zero.*
3. *Initial inventory is zero.*

Lemma 1.1 (Zero-inventory-ordering property)

Every order is received precisely when the inventory level drops to zero.

Definition 1.2 (Economic order quantity (EOQ))

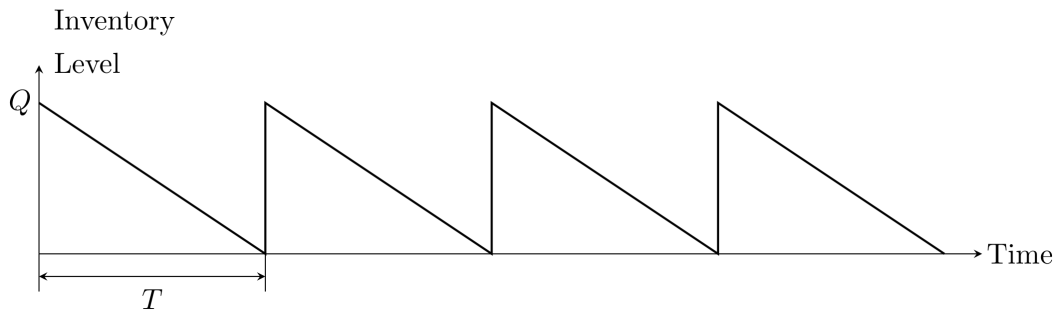
Given the objective to minimize the average total cost per unit of time (the total cost in a reorder interval is $K + h\int_0^T I(t)dt = K + \frac{hTQ}{2}$), by FOC, $Q^* = \sqrt{\frac{2KD}{h}}$.

$$\min_Q \frac{1}{T}\left(K + \frac{hTQ}{2}\right) = \frac{KD}{Q} + \frac{hQ}{2}$$

Remark EOQ is the quantity at which the ordering cost per unit of time (KD/Q) equals to the inventory holding cost per unit of time ($hQ/2$).

Note that these assumptions can be relaxed without losing generality,

1. If the order quantities cannot exceed C, then $Q^* = \min\{Q^*, C\}$
2. With lead time L, place Q^* when $I(t) = DL$.
3. With initial inventory I_0, then the first order is simply delayed until time I_0/D.



1.2 Power-of-Two Policies

Definition 1.3 (T^* **for Economic order quantity (EOQ))**

Given the objective to minimize the average total cost per unit of time, by FOC, $T^* = \frac{Q^*}{D} = \sqrt{\frac{2K}{hD}}$ *and* $f(T^*) = \sqrt{2KhD}$.

$$\min_T \frac{1}{T}\left(K + \frac{hTQ}{2}\right)$$

Definition 1.4 (Power-of-Two policy)

In this restriction, T *is restricted to be a power-of-two multiple of some fixed base planning period* T_B*, that is,* $T = T_B 2^k, \quad k \in \{0, 1, 2, \dots\}$.

Remark This policy makes T^* more implementable; otherwise, T^* may equal to $\sqrt{3}$ which is not implementable in practice.

Lemma 1.2

Under power-of-two policy, $k^* = \lceil \log_2(T^*/T_B) - 0.5 \rceil$ *and the average cost of the power-of-two policy is guaranteed to be within* 6% *of the overall policy.*

Proof

$$f\left(T_B 2^k\right) \le f\left(T_B 2^{k+1}\right) \quad k^* \text{ is the smallest integer satisfying it by convexity of } f$$

$$\frac{K}{T_B 2^k} + \frac{hD}{2} T_B 2^k \le \frac{K}{T_B 2^{k+1}} + \frac{hD}{2} T_B 2^{k+1}$$

$$\frac{K}{hD} \le \left(T_B 2^k\right)^2$$

$$\sqrt{\frac{K}{hD}} = \frac{1}{\sqrt{2}} T^* \le T_B 2^k$$

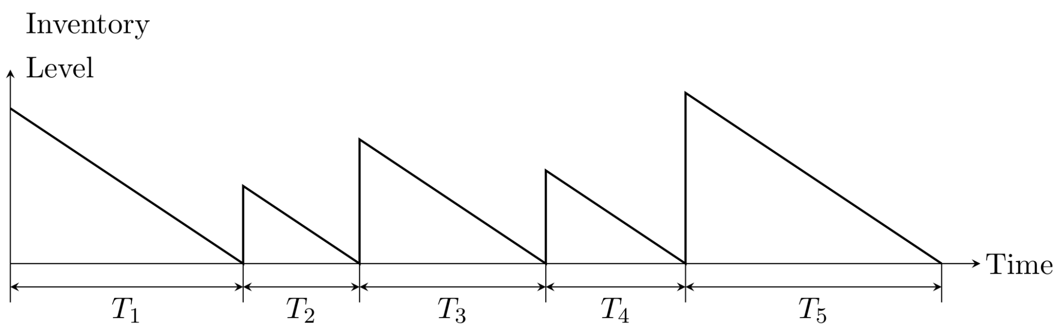
$$\log_2(T^*/T_B) - 0.5 \le k \le \log_2(T^*/T_B) + 0.5$$

Thus for any T_B, the optimal power-of-two policy must be in the interval $\left[T^*/\sqrt{2}, \sqrt{2}T^*\right]$ and $\frac{f(T)}{f(T^*)} \le 1.06$. ■

1.3 EOQ with finite horizon

1. $\tau = \sum_{i=1}^m T_i$: finite horizon

2. $\mathscr{P}$: inventory policy, places $m \geq 1$ orders in interval $[0, \tau]$, and relax the assumption that the order quantities are fixed.
3. T_i the time between the placement of ith order and $i+1$st order, T_m means the time between the placement of last order and τ



Lemma 1.3 (Optimal T^* given m)

Given m and minimize the total cost, $T^* = [T_1^*, \dots T_m^*]^\top = \frac{\tau}{m}$.

$$\min K_m + h \int_0^\tau I(t)dt \iff \min \int_0^\tau I(t)dt$$

Proof By zero-inventory-ordering property, we know $I(\tau) = 0$, thus our problem is divided into multiple segments.

$$\min \sum_{i=1}^m \frac{T_i \cdot T_i D}{2} = \frac{D}{2} \sum_{i=1}^m T_i^2$$

$$\min \left\{ \sum_{i=1}^m T_i^2 : \sum_{i=1}^m T_i = \tau, T_i \geq 0 \forall i = 1, \dots, m \right\}$$

$$\min \left\{ \sum_{i=1}^m T_i^2 : \sum_{i=1}^m T_i = \tau \right\} \quad \text{Relax}$$

By lagrangian $\mathcal{L}(\mathbf{T}, \lambda) = \sum_{i=1}^m T_i^2 - \lambda \left(\sum_{i=1}^m T_i - \tau \right)$ we can derive T^*. ■

Lemma 1.4

To minimize the total cost, $m^* = \tau \sqrt{\frac{hD}{2K}}$.

Proof By Lemma 1.3 we have $\min Km + \frac{hD\tau^2}{2m}$. ■

1.4 EOQ with backlogging demand

1.5 Economic production quantity (EPQ)

2 Inventory control by Dynamic Programming

2.1 Dynamic Programming

Definition 2.1 (Dynamic Programming)

1. k: *the index of time*
2. N: *the horizon of times*
3. x_k: *state of the system, an element of a space* S_k, *it summarizes the past information.*
4. u_k: *decision variable at time* k, *constrained to be in a subset* $V_k(x_k)$.
5. w_k: *random parameter/ disturbance/ noise, characterized by* $P(\cdot \mid x_k, u_k)$, *but does not depend on prior disturbances* $w_{k-1}, \ldots, w_0$. *The system is deterministic if each* w_k *can take only one value.*
6. $x_{k+1} = f_k(x_k, u_k, w_k), \quad k = 0, 1, \ldots, N-1$: *state transition equation*
7. $g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)$ *or* $E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)\right\}$: *objective function*

Definition 2.2 (Inventory control by DP)

1. x_k: *stocks available at the beginning of the kth period*
2. u_k: *stocks ordered at the beginning of the kth period*
3. w_k: *demand during the kth period, assume the excess demand is backlogged*
4. $x_{k+1} = x_k + u_k - w_k, \quad k = 0, 1, \ldots, N-1$: *state transition equation*
5. $h(x_k)$: *include holding cost for positive stock and shortage cost for negative stock*
6. $c(u_k)$: *purchasing cost*
7. $\min_{u_i \geq 0} E\left\{g_N(X_N) + \sum_{k=0}^{N-1}(h(x_{k+1}) + c(u_k))\right\}$, *where* $g_N(X_N)$ *is the terminal cost*

Definition 2.3 (Open,Close loop optimization)

1. *Open-loop optimization means select all decisions* $u_0, \ldots, u_{N-1}$ *at one at time 0*
2. *Closed-loop optimization means postpone the decision* u_k *until* x_k *is known (excess information).*

Remark In closed-loop, we are not interested in finding optimal numerical values of u_k, but rather we want to find an optimal rule/ policy $\mu_k(x_k)$.

Definition 2.4 (Policy)

1. $\mu_k(x_k)$: *the action to be taken at time* k *if the state is* x_k
2. $\pi = \{\mu_0, \ldots, \mu_{N-1}\}$: *a policy or control law*

3. *Admissible policy: A policy such that* $\mu_k(x_k) \in U_k(x_k)\, \forall x_k \in S_k$
4. $x_{k+1} = f_k(x_k, \mu_k(x_k), w_k), \quad k = 0, 1, \ldots, N-1$*: state transition equation*
5. $J_\pi(x_0) = E\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, \mu_k(x_k), w_k)\right\}$*: objective function*
6. $J^*(x_0) = \min_{\pi\in\Pi} J_\pi(x_0)$*: Optimal value*
7. $J_{\pi^*}(x_0) = \min_{\pi\in\Pi} J_\pi(x_0)$*: Optimal policy*

Theorem 2.1 (Principle of Optimality)

If a policy $\{\mu_0^*, \mu_1^*, \ldots, \mu_{N-1}^*\}$ *is optimal for the problem from time 0 to time N, then the truncated policy* $\{\mu_k^*, \mu_{k+1}^*, \ldots, \mu_{N-1}^*\}$ *is optimal for the subproblem minimizing the cost from time* k *to time N.*

Remark The tail portion of an optimal policy is optimal for the tail subproblem.

Theorem 2.2 (DP is optimal)

For every initial state x_0*, the optimal cost* $J^*(x_0)$ *is equal to* $J_0(x_0)$ *by DP:*

$$J_N(x_N) = g_N(x_N)$$

$$J_k(x_k) = \min_{u_k\in U_k(x_k)} E_{w_k}\{g_k(x_k, u_k, w_k) + J_{k+1}(f_k(x_k, u_k, w_k))\}, \quad k = 0, 1, \ldots, N-1,$$

Furthermore, if $u_k^* = \mu_k^*(x_k)$ *minimizes this DP, then the policy* $\pi^* = \{\mu_0^*, \ldots, \mu_{N-1}^*\}$ *is optimal.*

Proof ■

2.2 Asset Selling

Definition 2.5

1. T*: termination state*
2. r*: if accept the offer, he can invest the money at a fixed rate of interest* $r > 0$
3. x_K*: if* $x_k = T \forall k \le T-1$*, say the asset has been sold*
4. $\{u', u^2\}$*: control space, means { "sell", "not sell" }, no more decisions if the asset has been sold in the kth stage.*
5. w_k*: disturbance at time* k*, is i.i.d Random variable*
6. *state transition equation*

$$x_{k+1} = f_k(x_k, u_k, w_k) = \begin{cases} T & \text{if } x_k = T, \text{ or if } x_k \neq T \text{ and } u_k = u^1 \text{ (sell)} \\ w_k & \text{otherwise} \end{cases} \quad k = 1, \ldots, N-1$$

7. *Objective function:* $E_{w_0,\dots,w_{N-1}}\left\{g_N(x_N)+\sum_{k=1}^{N-1}g_k(x_k,u_k,w_k)\right\}$

$$g_N(x_N)=\begin{cases}x_N & \text{if } x_N\neq T\\ 0 & \text{otherwise}\end{cases}$$

$$g_k(x_k,u_k,w_k)=\begin{cases}(1+r)^{N-k}x_k & \text{if } x_k\neq T \text{ and } u_k=u^1(\text{ sell })\\ 0 & \text{otherwise}\end{cases}$$

Lemma 2.1

With DP

$$J_N(x_N)=\begin{cases}x_N & \text{if } x_N\neq T,\\ 0 & \text{otherwise },\end{cases}$$

$$J_k(x_k)=\begin{cases}\max\left[(1+r)^{N-k}x_k, E\left\{J_{k+1}(w_k)\right\}\right] & \text{if } x_k\neq T\\ 0 & \text{if } x_k=T\end{cases}$$

Let $\alpha_k=\frac{E\{J_{k+1}(w_k)\}}{(1+r)^{N-k}}$*, the optimal policy is*

1. *Accept the offer* x_k *if* $x_k>\alpha_k$
2. *Reject the offer* x_k *if* $x_k<\alpha_k$

Proof Firstly we should prove $\alpha_{k-1}\geqslant\alpha_k$, it means that if an offer is good enough to be acceptable at time $k-1$, it should also be acceptable at time k. Note that prove $J_k(x)\geq(1+r)J_{k+1}(x)\quad\forall x\neq T$ is enough, and it can be proved by induction. ■

2.3 Dynamic Lot-Sizing

Consider we want to plan a sequence of orders over T periods, keep 3 assumptions

1. d_t: demand at period t, deterministic
2. K: fixed order cost for every order; c: per unit order cost. Thus if ordering z units, order cost is

$$cz+KI_{z>0}\quad I_a=1,0$$

3. h: holding cost per unit per period
4. objective function

$$\begin{aligned}\min\quad & \sum_{t=1}^{T}\left[K\mathbb{I}_{z_t>0}+hI_t\right]\\ \text{s.t.}\quad & I_t=I_{t-1}+z_t-d_t,\quad t=1,\dots,T\quad\text{(Inventory-balance constraint)}\\ & I_0=0\quad\text{(Initial inventory)}\\ & I_t,z_t\geq 0,\quad t=1,\dots,T\end{aligned}$$

Lemma 2.2 (Zero-inventory-ordering property)

Any optimal policy is a zero-inventory ordering policy, i.e., a policy in which

$$z_t I_{t-1} = 0, \quad \text{for } t = 1, \ldots, T$$

Remark A simple corollary is that in an optimal policy an order is of size equal to satisfy demand for an integer number of subsequent periods. So the problem can be transferred to the decision of time to order.

1. n_i: state, means the 1st time to place an order in periods $\{i, \ldots, T+1\}$, $n_i = T+1$ means no order from i to T
2. u_i: control variable, means the time for 1st order in $\{i+1, \ldots, T+1\}$, must be chosen from $U_i(n_i)$

$$U_i(n_i) = \begin{cases} \{i+1, \ldots, T+1\} & \text{if } n_i = i \\ n_i & \text{if } n_i > i \end{cases}$$

3 Stochastic Newsvendor

3.1 Single Period Newsvendor

Definition 3.1

Assume demand D with $F(.)$, unit selling price r, unit cost c and salvage value v ($r > c > v$), let y denote the amount produced, then we want to minimize the expected cost

$$\min_y f(y) = cy - rE[\min\{y, D\}] - vE\left[(y-D)^+\right]$$

Here

$$a^+ = \max\{0, a\} \quad \min\{y, D\} = D - (D-y)^+ \quad (D-y)^+ - (y-D)^+ = D - y$$

And optimal $y^* = S$ satisfy $F(S) = \frac{r-c}{r-v}$.

Remark Optimality means the balance between the cost of being understocked and the total costs of being either overstocked or understocked.

$$\frac{r-c}{r-v} = \frac{\text{underage cost}}{\text{overage cost } + \text{ underage cost}}$$

Proof

$$\begin{aligned} f(y) &= cy - rE[D] + rE\left[(D-y)^+\right] - vE\left[(y-D)^+\right] \\ &= cy - rE[D] + (r-v)E\left[(D-y)^+\right] + vE[D-y] \\ &= (c-v)y - (r-v)E[D] + (r-v)\int_u^\infty (D-y)dF(D). \end{aligned}$$

Take the FOC we have $(c-v) - (r-v)(1-F(y)) = 0$. ■

Definition 3.2 (**(s, S) policy**)

Assume initial inventory is x and a fixed set-up cost K, then order $S - x$ if $x \leq s$, otherwise do not order.

Definition 3.3 (Discrete Newsvendor's Optimal)

Suppose demand can be $D_1, D_2, \ldots, D_n$ with probability $p_1, p_2, \ldots, p_n$, then the optimal order quantity must be one of the demand points, $D_1, D_2, \ldots, D_n$.

3.2 Multiple Period (Finite) Newsvendor

Definition 3.4 (DP for Multiple Period Newsvendor)

Consider T periods, the inventory level at the beginning of t period is x_t, the inventory level at the end of t period is y_t, the demand for period t is D_t (iid). If $D_t \geq y_t$, then the additional demand is backlogged to the next period, thus we have negative inventory, e.g. $x_{t+1} = y_t - D_t$.

Ordering cost consists of a set-up cost K, a proportional purchase cost c, the ordering cost is $K\mathbb{I}_{y_t > x_t} + c(y_t - x_t)$. Holding cost h and Shortage cost b means the expected one-period shortage and holding cost is $H_t(y_t) = hE\left[(y_t - D_t)^+\right] + bE\left[(D_t - y_t)^+\right]$, note that it is convex.

Let $H_{T+1}(x_{T+1}) = -cx_{T+1}$ as the boundard condition. Then we have a DP:

$$\begin{aligned} J_t(x_t) &= \min_{y_t \geq x_t} \left\{K\mathbb{I}_{y_t > x_t} + c(y_t - x_t) + H_t(y_t) + E\left[J_{t+1}(y_t - D_t)\right]\right\} \quad \forall t = 1, \ldots, T \\ &= -cx_t + \min_{y_t \geq x_t} \left\{K\mathbb{I}_{y_t > x_t} + f_t(y_t)\right\} \quad (f_t(y_t) = cy_t + H_t(y_t) + E\left[J_{t+1}(y_t - D_t)\right]) \end{aligned}$$

$J_{T+1}(x_{T+1}) = H_{T+1}(x_{T+1})$

Definition 3.5 (K-Convex Function)

A real valued function f is called K-convex for $K \geq 0$ if for any $x_0 \leq x_1$ and $\lambda \in [0, 1]$.

$$f((1 - \lambda)x_0 + \lambda x_1) \leq (1 - \lambda)f(x_0) + \lambda f(x_1) + \lambda K$$

Lemma 3.1 (Properties)

1. *A real-valued convex function is also 0-convex and hence K-convex for all $K \geq 0$. A K_1-convex function is also a K_2-convex function for $K_1 \leq K_2$.*
2. *If $f_1(y)$ and $f_2(y)$ are K_1-convex and K_2-convex, respectively, then for $\alpha, \beta \geq 0$, $\alpha f_1(y) + \beta f_2(y)$ is $(\alpha K_1 + \beta K_2)$-convex.*
3. *If $f(y)$ is K-convex and ζ is a random variable, then $E_\zeta[f(y - \zeta)]$ is also K-convex, provided $E[|f(y - \zeta)|] < \infty$ for all y.*

Proposition 3.1

Assume that f is a continuous K-convex function for some $K > 0$ and $f(y) \to \infty$ as $|y| \to \infty$. Let S be a minimum point of f and s be any element of the set $\{x : x \leq S, f(x) = f(S) + K\}$, then

1. $f(y) \geq f(s) = f(S) + K$ *for all* $y \leq s$
2. $f(y)$ *is a non-increasing function on* $(-\infty, s)$
3. $f(y) \leq f(z) + K$ *for all* y, z *such that* $s \leq y \leq z$

Proposition 3.2

If $f(x)$ is a K-convex function, then $g(x) = \min_{y \geq x} \{Q\mathbb{I}_{y>x} + f(y)\}$ is $\max\{K, Q\}$-convex.

Theorem 3.1 (Optimalit of (s_t, S_t) Policy)

1. *For any $t = 1, ..., T$, $f_t(y)$ and $J_t(x)$ are continuous and $\lim_{|y| \to \infty} f_t(y) = \infty$*
2. *For any $t = 1, ..., T$, $f_t(y)$ and $J_t(x)$ are K-convex.*
3. *For any $t = 1, ..., T$, there exist two parameters s_t and S_t such that it is optimal to make an order to raise the inventory to S_t when the initial inventory level is no more than s_t and to order nothing otherwise.*

Definition 3.6 (Multiple period Newsvendor with Leadtime L)

3.3 Integration of Inventory and Pricing

Definition 3.7

Consider we also decide the selling price and the demand ξ_t depends on p_t, $\xi_t = \alpha_t D_t(p_t) + \beta_t$, here

1. $\alpha_t \geq 0, E[\alpha_t] = 1$
2. $E[\beta_t] = 0$
3. *$D_t(p_t)$ is continuous and strictly decreasing for any $p_t \in \left[\underline{p}_t, \bar{p}_t\right]$, since $E[\alpha_t] = 1, E[\beta_t] = 0$, $D_t(p_t)$ can be interpreted as the expected demand for p_t*

For (α_t, β_t),

1. *If α_t is deterministic, this demand function is the additive demand function, i.e. $\xi_t = D_t(p_t) + \beta_t$.*
2. *If β_t is deterministic, it is multiplicative demand function, i.e. $\xi_t = \alpha_t D_t(p_t)$*

Assume that lead time is zero and unsatisfied demand is backlogged, and let x_t and y_t be the inventory levels at the beginning and the end of period t, then

$$x_{t+1} = y_t - \xi_t = y_t - \alpha_t D_t(p_t) - \beta_t$$

Assume no fixed set-up cost, the ordering cost is $c(y_t - x_t)$, let $h(x_{t+1})$ denotes the *holding or shortage cost, then the expected revenue is* $E\left[p_t\xi_t\right] = p_t D_t\left(p_t\right)$*, and the expected cost is* $p_t D_t\left(p_t\right) - c\left(y_t - x_t\right) - E\left[h\left(y_t - \alpha_t D_t\left(p_t\right) - \beta_t\right)\right]$*. We have a DP*

$$J_t\left(x_t\right) = \max_{\substack{y_t \geq x_t, \\ p_t \in \left[\underline{p}_t, \bar{p}_t\right]}} \left\{p_t D_t\left(p_t\right) - c\left(y_t - x_t\right) - E\left[h\left(y_t - \alpha_t D_t\left(p_t\right) - \beta_t\right)\right] + E\left[J_{t+1}\left(y_t - \alpha_t D_t\left(p_t\right) - \beta_t\right.\right.\right.$$

$$= \max_{\substack{y_t \geq x_t, \\ d_t \in \left[\underline{d}_t, \bar{d}_t\right]}} \left\{R_t\left(d_t\right) - c\left(y_t - x_t\right) - E\left[h\left(y_t - \alpha_t d_t - \beta_t\right)\right] + E\left[J_{t+1}\left(y_t - \alpha_t d_t - \beta_t\right)\right]\right\} \quad (d_t =$$

$$= c x_t + \max_{y_t \geq x_t} f_t\left(y_t\right) \quad \left(f_t\left(y_t\right) = \max_{d_t \in \left[\underline{d}_t, \bar{d}_t\right]} \left\{R_t\left(d_t\right) - E\left[h\left(y_t - \alpha_t d_t - \beta_t\right)\right] + E\left[J_{t+1}\left(y_t - \alpha_t d\right.\right.\right.\right.$$

$$J_{T+1}\left(x_{T+1}\right) = c x_{T+1}$$

Note that $R_t(d_t)$ *is concave and* $h(x_{t+1})$ *is convex.*

Definition 3.8 (Base stock list price policy)

Definition 3.9 (Join)

$$\mathbf{x} \vee \mathbf{x}' = \left(\max\left\{x_1, x_1'\right\}, \max\left\{x_2, x_2'\right\}, \ldots, \max\left\{x_n, x_n'\right\}\right)$$

Definition 3.10 (Meet)

$$\mathbf{x} \wedge \mathbf{x}' = \left(\min\left\{x_1, x_1'\right\}, \min\left\{x_2, x_2'\right\}, \ldots, \min\left\{x_n, x_n'\right\}\right)$$

Definition 3.11 (Supermodular)

Consider a function $f : X \mapsto \mathbb{R}$*, where* $X \subseteq \mathbb{R}^n$*. The function* f *is supermodular on the set* X*, if for any* $\mathbf{x}, \mathbf{x}' \in X$*,*

$$f(\mathbf{x}) + f\left(\mathbf{x}'\right) \leq f\left(\mathbf{x} \vee \mathbf{x}'\right) + f\left(\mathbf{x} \wedge \mathbf{x}'\right)$$

whenever $\mathbf{x} \vee \mathbf{x}', \mathbf{x} \wedge \mathbf{x}' \in X$*.*

Proposition 3.3

1. *Any positively linear combination of supermodular functions is supermodular.*
2. *Assume that a function* $f(.,.)$ *is defined in the product space* $\mathbb{R}^n \times \mathbb{R}^m$*. If* $f(.,y)$ *is supermodular for any given* $\mathbf{y} \in \mathbb{R}^m$*, then for a random vector* $\boldsymbol{\zeta} \in \mathbb{R}^m$*,* $E_\zeta[f(\mathbf{x}, \boldsymbol{\zeta})]$ *is supermodular, provided it is well defined.*

Proposition 3.4

For $a_1, a_2 \geq 0$ *and a concave function* $f : \mathbb{R} \mapsto \mathbb{R}, g\left(x_1, x_2\right) = f\left(a_1 x_1 - a_2 x_2\right)$ *is supermodular.*